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UNIFORMLY LOADED, SIMPLY SUPPORTED, ANTISYMMETRICALLY LAMINATED, RECTANGULAR PLATE ON A WINKLER–PASTERNAK FOUNDATION

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Abstract – An analysis of a uniformly loaded, simply supported, antisymmetrically laminated, cross- and angle-ply, rectangular plate resting on a general (Winkler–Pasternak) elastic foundation is presented. A computer program has been written to evaluate the important results: deflections, stress couples etc., of the analysis. Typical results for a number of different plate lay-ups and Winkler–Pasternak values are presented in dimensionless graphical form. These serve to illustrate the importance of material coupling and foundation stiffness on the plate response.

NOTATION

a	plate length
$A_{11}, A_{12}, A_{22}, A_{66}$	extensional plate stiffnesses
b	plate width
B_{11}, B_{16}, B_{26}	coupling plate stiffnesses
$D_{11}, D_{12}, D_{22}, D_{66}$	flexural plate stiffnesses
E_L, E_T	longitudinal and transverse elastic moduli
$E_{mn}, F_{mn}, G_{mn}, q_{mn}$	displacement and lateral load amplitude coefficients
g	pasternak foundation stiffness
$G(=gb^2/E_Th_0^3)$	dimensionless Pasternak foundation stiffness
G_{LT}	shear modulus
h_0	plate thickness
k	Winkler foundation stiffness
$K(=kb^4/E_Th_0^3)$	dimensionless Winkler foundation stiffness
N_x, N_y, N_{xy}	stress resultants
M_x, M_y, M_{xy}	stress couples
q, q_0	lateral load
$Q(=q_0b^4/E_Th_0^4)$	dimensionless lateral load
$R(=a/b)$	plate aspect ratio
θ	lamina orientation angle relative to the longitudinal co-ordinate
u, v	displacements
ν_{LT}	Poisson's ratio
w	deflection
x, y	longitudinal and transverse co-ordinates
$(\ )^{\cdot}$	$\partial(\ )/\partial x$
$(\ )'$	$\partial(\ )/\partial y$.

INTRODUCTION

Since the appearance of Reissner and Stavsky's paper [1], in which material coupling effects in laminated plates were properly accounted for, many papers have been published dealing with these phenomena. Only recently, however, have structural elements been considered in which coupling effects are combined with the presence of an elastic foundation. In a recent paper [2], the author presented simple, exact solutions for uniformly loaded, simply supported, antisymmetrically laminated cross- and angle-ply plate strips on Winlker–Pasternak foundations. This paper represents a first step in the consideration of finite plate behaviour, and as such is a logical sequal to the earlier work. Both antisymmetrically laminated, cross- and angle-ply plates resting on a Winkler–Pasternak foundation are considered.

The main purpose of this paper is to present non-dimensional results, which indicate quantitatively the influence of the foundation and material coupling on the plate response, when it is uniformly loaded. However, because of the large number of parameters, which characterise the problem, it is not possible to present universally applicable results, or even a complete set of results. Consequently, all the results presented herein are restricted to laminated plates fabricated from Carbon Fibre Reinforced Plastic (CFRP). Results for other materials are, however, readily generated by the computer program, which has been used to evaluate the analysis presented in a later section of the paper.

MATERIAL PROPERTIES AND LAY-UP

In deriving the results, which are plotted in Figs. 1–10 typical values for the elastic constants of CFRP have been used and they are,

$$E_L/E_T = 40, \quad G_{LT}/E_T = 0.5 \quad \text{and} \quad \nu_{LT} = 0.25.$$

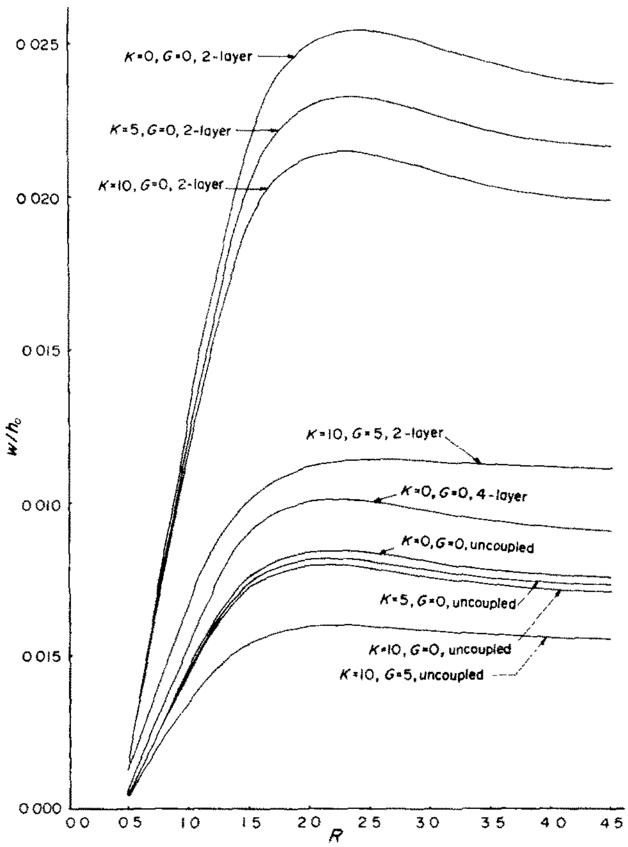

Fig. 1. Plate centre deflection–plate aspect ratio curves for an antisymmetrically laminated, cross-ply plate ($Q = 1$).

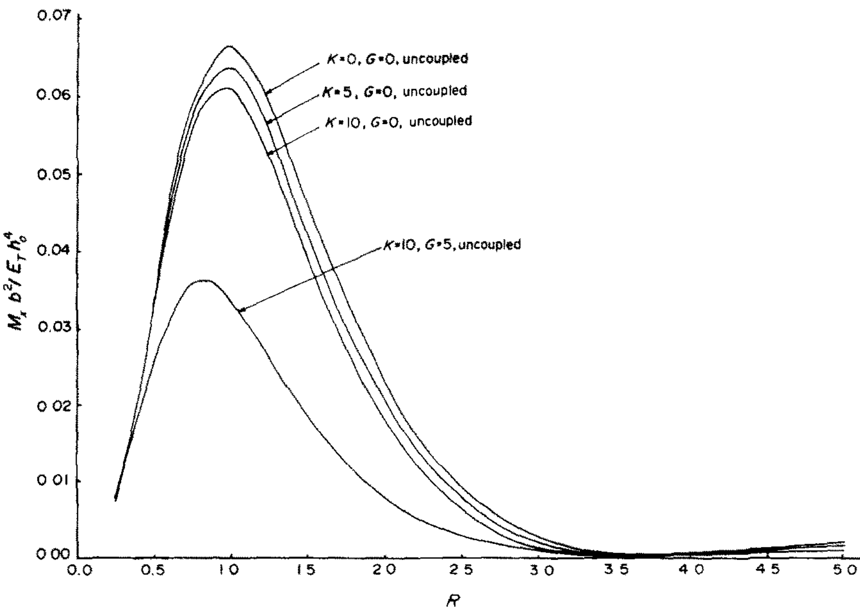

Fig. 2. Plate centre stress couple–plate aspect ratio curves for an antisymmetrically laminated, cross-ply plate ($Q = 1$).

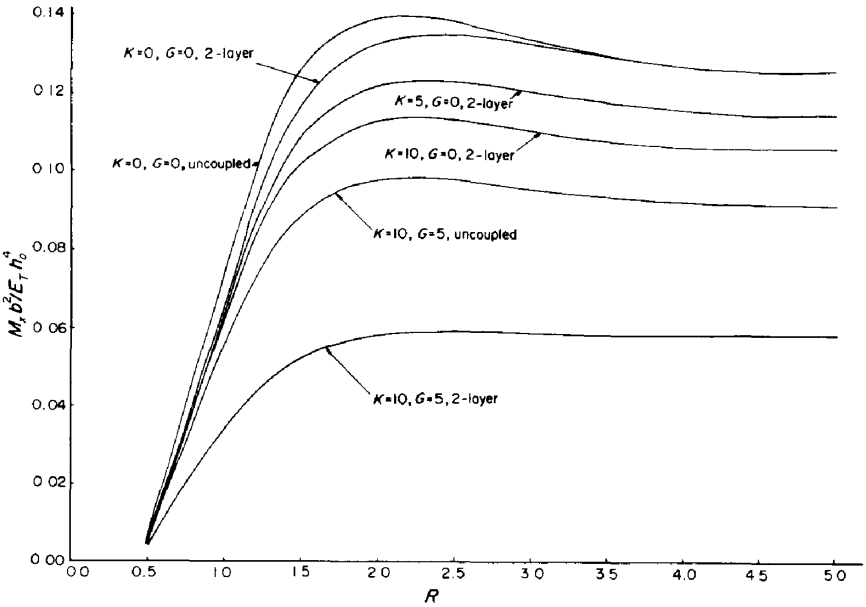

Fig. 3. Plate centre stress couple–plate aspect ratio curves for an antisymmetrically laminated, cross-ply plate ($Q = 1$).

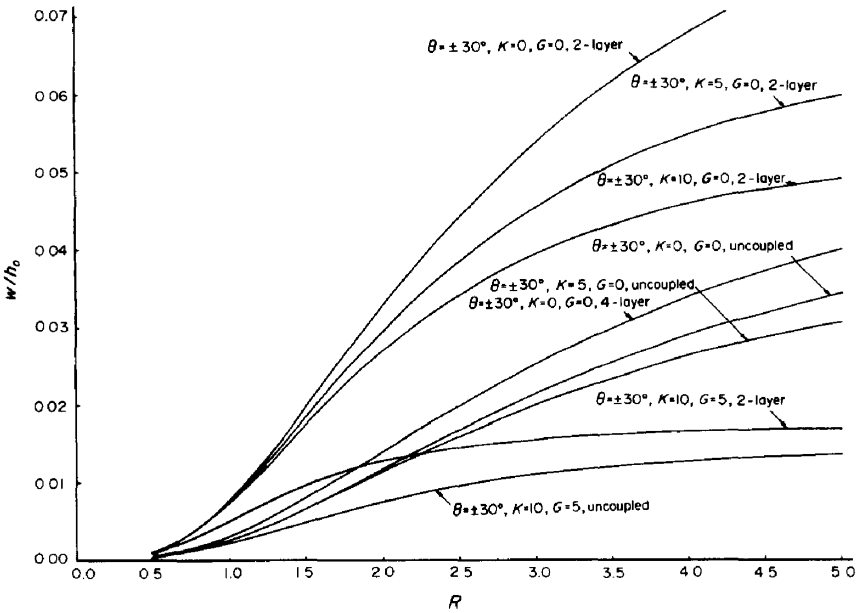

Fig. 4. Plate centre deflection–plate aspect ratio curves for an antisymmetrically laminated, angle-ply plate ($\theta = \pm 30°$, $Q = 1$).

Two basic types of plate lay-up are considered: antisymmetric cross-ply and antisymmetric angle-ply. In the former type of plate, the lamina principal axes are alternately aligned with and at right angles to the plate axes, whereas in the latter, they are inclined at $+\theta$ and $-\theta$ to the plate axes. Each plate is designated according to the number of constituent laminae as: 2-layer, 4-layer or uncoupled. These labels imply that the lamina thicknesses in the first two instances are respectively, $\frac{1}{2}h_0$ and $\frac{1}{4}h_0$ and in the third instance that the plate is homogeneous rather than laminated.

ANALYSIS

(a) *Antisymmetric cross-ply plate*

The equilibrium equations for this type of plate are obtained from those of Whitney [3] by

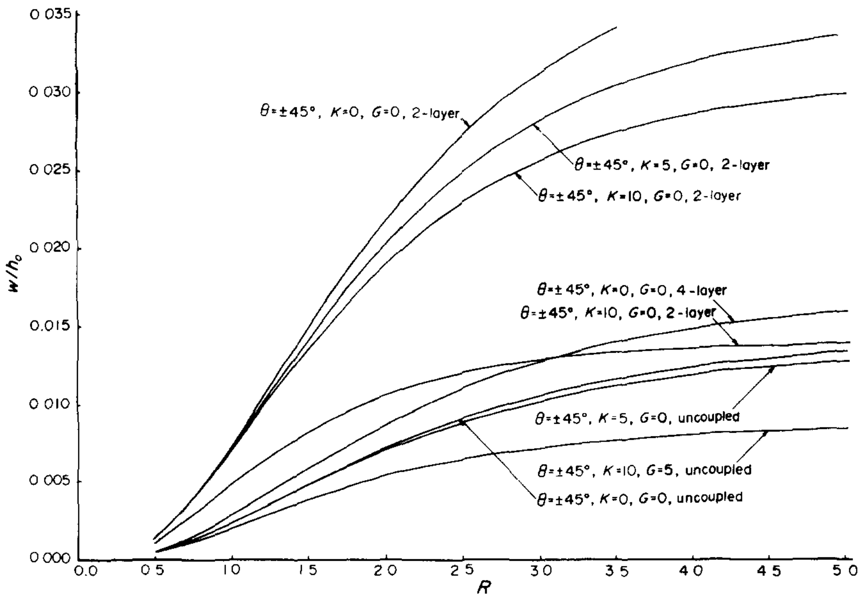

Fig. 5. Plate centre deflection–plate aspect ratio curves for an antisymmetrically laminated, angle-ply plate ($\theta = \pm 45°$, $Q = 1$).

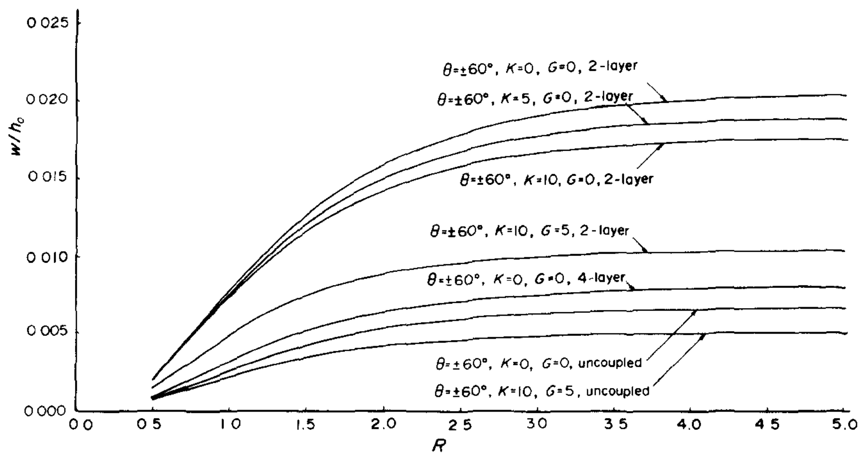

Fig. 6. Plate centre deflection–plate aspect ratio curves for an antisymmetrically laminated, angle-ply plate ($\theta = \pm 60°$, $Q = 1$).

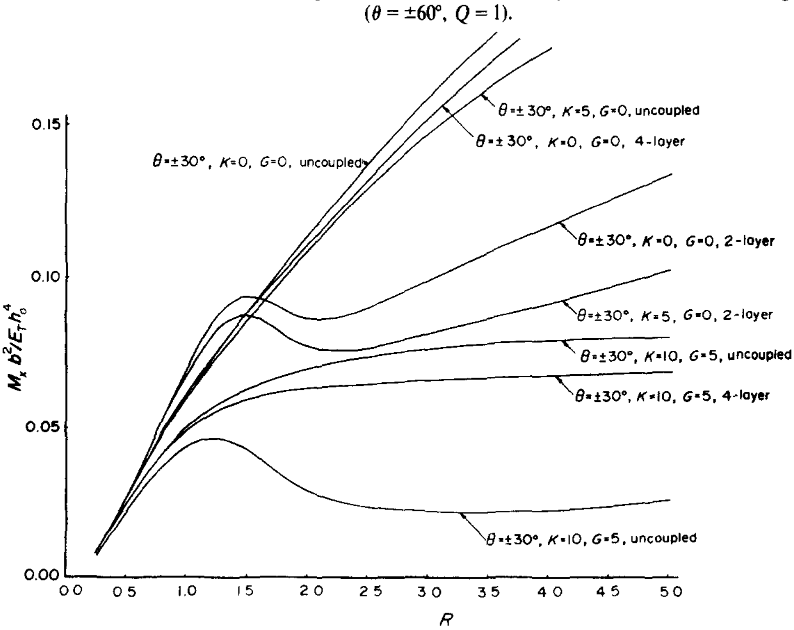

Fig. 7. Plate centre stress couple–plate aspect ratio curves for an antisymmetrically laminated, angle-ply plate ($\theta = \pm 30°$, $Q = 1$).

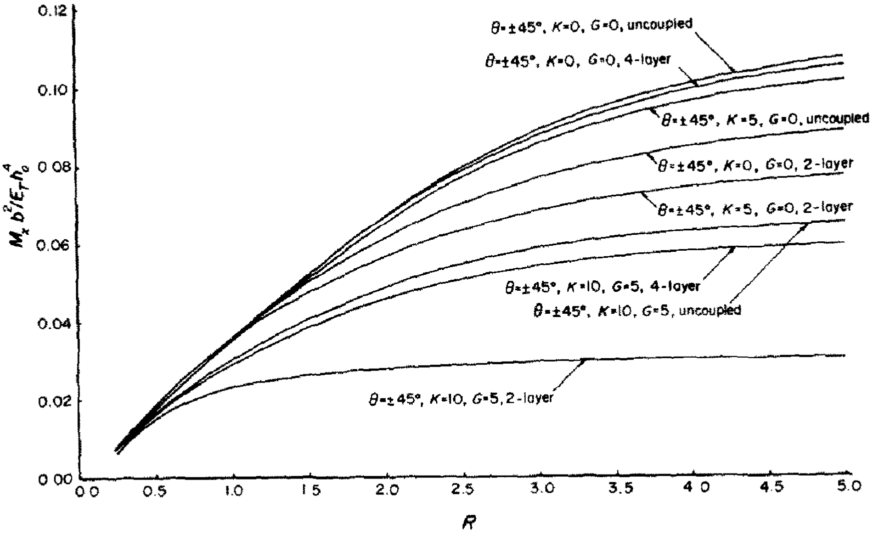

Fig. 8. Plate centre stress couple–plate aspect ratio curves for an antisymmetrically laminated, angle-ply plate ($\theta = \pm 45°$, $Q = 1$).

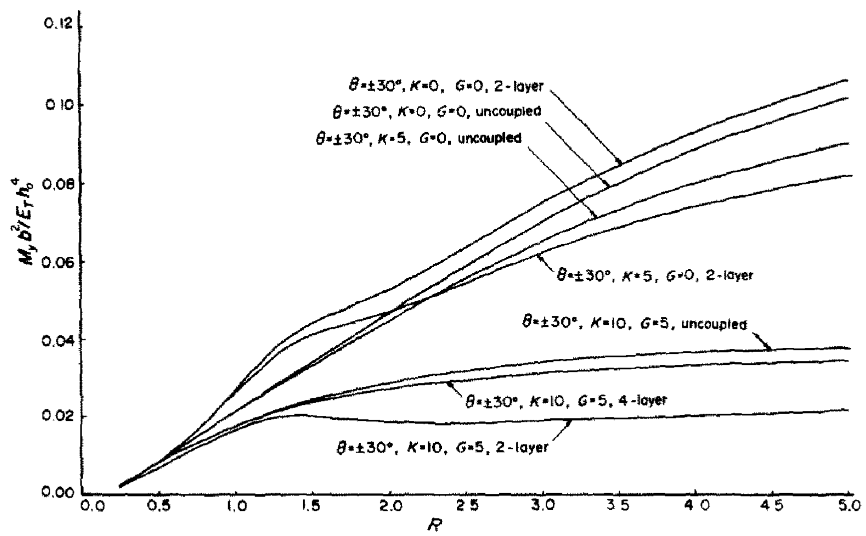

Fig. 9. Plate centre stress couple–plate aspect ratio curves for an antisymmetrically laminated, angle-ply plate ($\theta = \pm 30°$, $Q = 1$).

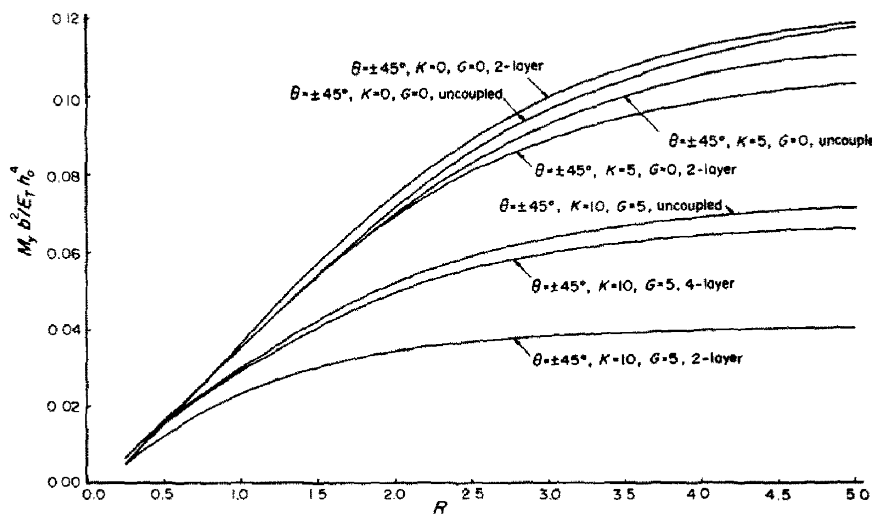

Fig. 10. Plate centre stress couple–plate aspect ratio curves for an antisymmetrically laminated, angle-ply plate ($\theta = \pm 45°$, $Q = 1$).

adding two terms, corresponding to the Winkler–Pasternak foundation reaction, to the right hand side of the out-of-plane equilibrium equation so that,

$$
\begin{aligned}
&A_{11}u^{\cdot\cdot} + A_{66}u'' + (A_{12} + A_{66})v^{\cdot\prime} - B_{11}w^{\cdot\cdot\cdot} = 0 \\
&(A_{12} + A_{66})u^{\cdot\prime} + A_{66}v^{\cdot\cdot} + A_{11}v'' + B_{11}w''' = 0 \qquad (1) \\
&D_{11}(w^{\cdot\cdot\cdot\cdot} + w'''') + 2(D_{12} + 2D_{22})w^{\cdot\cdot\prime\prime} - V_{11}(u^{\cdot\cdot\cdot} - v''') = q - kw + g(w^{\cdot\cdot} + w'')
\end{aligned}
$$

A navier-type solution is postulated, so that the following forms for the displacement components are appropriate,

$$u = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} E_{mn} \cos(m\pi x/a) \sin(n\pi y/b)$$

$$v = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} F_{mn} \sin(m\pi x/a) \cos(n\pi y/b). \qquad (2)$$

and

$$w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} G_{mn} \sin(m\pi x/a) \sin(n\pi y/b)$$

If the lateral load, q, is postulated as follows,

$$q = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} q_{mn} \sin(m\pi x/a) \sin(n\pi y/b) \qquad (3)$$

then eqns (2) and (3) may be substituted into eqns (1) to yield a set of three simultaneous equations, which are readily solved for the unknown coefficients, A_{mn} etc. Substituting these latter into eqns (2) and noting that for a uniformly loaded plate,

$$q_{mn} = 16q_0(\pi^2 mn)^{-1} \quad \text{where} \quad m, n = 1, 3, 5 \quad \text{etc.}$$

the expressions for the plate displacements become fully defined. Explicit expressions for the stress resultants and stress couples are obtained by substituting the derivatives of the fully defined forms of eqns (2) into the constitutive equations[3].

The above expressions for the displacements imply the following boundary conditions,

(i) *Along* $x = 0$, *Rb*

$$u \neq 0 \quad \text{and} \quad v = w = 0$$

$$N_x = N_y = 0, \quad N_{xy} \neq 0, \quad M_x = M_y = 0 \quad \text{and} \quad M_{xy} \neq 0.$$

(ii) *Along* $y = 0$, b

$$u = 0, \quad v \neq 0 \quad \text{and} \quad w = 0$$

$$N_x = N_Y = 0, \quad N_{xy} \neq 0, \quad M_x = M_Y = 0 \quad \text{and} \quad M_{xy} \neq 0.$$

(b) *Antisymmetric angle-ply plate*

The equilibrium equations are again obtained by modifying those given by Whitney[3], so that,

$$
\begin{aligned}
&A_{11}u^{\cdot\cdot} + A_{66}u'' + (A_{12} + A_{66})v^{\cdot\prime} - 3B_{16}w^{\cdot\cdot\prime} - B_{26}w''' = 0 \\
&(A_{12} + A_{66})u^{\cdot\prime} + A_{66}v^{\cdot\cdot} + A_{22}v'' - B_{16}w^{\cdot\cdot\cdot} - 3B_{26}w^{\cdot\prime\prime} = 0 \qquad (4) \\
&D_{11}w^{\cdot\cdot\cdot\cdot} + 2(D_{12} + 2D_{66})w^{\cdot\cdot\prime\prime} + D_{22}w'''' - 3B_{16}u^{\cdot\cdot\prime} - B_{26}u''' - B_{16}v^{\cdot\cdot\cdot} - 3B_{26}v^{\cdot\prime\prime} = q - kw + g(w^{\cdot\cdot} + w'')
\end{aligned}
$$

Suitable displacement functions are obtained by replacing sine by cosine and vice versa in the first two of eqns (2). Substituting these plus eqn (3) into eqns (4) and solving, fully defined expressions for the displacements may be determined. Explicit expressions for the stress resultants etc., may then be obtained from the appropriate constitutive eqns[3].

In this case, the implied boundary conditions are,

(i) *Along* $x = 0$, Rb

$$u = 0, \quad v \neq 0 \quad \text{and} \quad w = 0$$

$$N_x \neq 0, \quad N_y \neq 0, \quad N_{xy} = M_x = M_y = 0 \quad \text{and} \quad M_{xy} \neq 0$$

(ii) *Along* $y = 0$, b

$$u \neq 0, \quad v = 0 \quad \text{and} \quad w = 0$$

$$N_x \neq 0, \quad N_y \neq 0, N_{xy} = M_x = M_y = 0 \quad \text{and} \quad M_{xy} \neq 0$$

RESULTS

A computer program has been written to evaluate the displacements and stress resultants/couples for both cross- and angle-ply plates. The initial results obtained from the program were for a number of special cases[2, 4]. Details of these results are not presented, but they serve to confirm that the computer program functions satisfactorily and may be used to provide quantitative information about the plate response in the presence of a Winkler–Pasternak foundation. However, because of the large number of parameters defining the problem, a complete set of results is not possible. Hence, only selected results are presented in Figs. 1–10.

The cross-ply plate results, which are more comprehensive than those for the angle-ply plate, are presented in Figs. 1–3. They show the variation of the deflection, w, and the two stress couples, M_x and M_y, at the plate centre with the plate aspect ratio, R. All these quantities are sensibly constant and independent of the plate aspect ratio when $R > 4$. Furthermore, these results indicate that a change in the value of the Pasternak foundation stiffness exerts a greater influence on the plate response, i.e. changes w, M_x or M_y more, than does an equal change in the value of the Winkler foundation stiffness.

A rather limited selection of angle-ply plate results is presented in Figs. 4–10. Figures 4–6 show the variation of the central deflection with the plate aspect ratio for ±30°, ±45° and ±60° lay-ups respectively. Here too, the deflections are constant and independent of the plate aspect ratio when R is large.

Figures 7 and 8 depict M_x as a function of R for ±30° and ±45° plates respectively. In Fig. 7, the shape of the $M_x - R$ curve for 2-layer plates is significantly different from those of the 4-layer and uncoupled plates. This type of behaviour also occurs with ±15° and ±60° lay-ups. It also appears that for small R values, M_x may be greatest in 2-layer plates. This is especially so when K and G are small.

In Figs. 9 and 10, M_y is presented as a function of R for ±30° and ±45° plates respectively. Again, the shape of the $M_y - R$ curve for 2-layer plates differs from those of the 4-layer and uncoupled plates, but rather less markedly.

It is evident from Figs. 1–10 that both material coupling and the foundation stiffnesses significantly affect the plate response for most aspect ratios.

CONCLUSIONS

A computer program has been written to evaluate displacements etc. for uniformly loaded, simply supported, astisymmetrically laminated, cross- and angle-ply, rectangular plates resting on a Winkler–Pasternak foundation. The program has been verified for a number of special cases. It has then been used to provide a limited set of results, which give some insight into the behaviour of such plates. From a consideration of these, it appears that material coupling significantly affects the plate response even in the presence of a general, elastic foundation. Furthermore, it is evident that a change in the Pasternak foundation stiffness produces a much greater change in the plate response than a similar change in the Winkler foundation stiffness.

Finally, the results presented serve to illustrate the need to carry out a proper analysis, since the variation of the stress couples, in particular, for some lay-ups is not intuitively obvious.

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